

# SHIFTED CONVOLUTION $L$ -SERIES VALUES FOR ELLIPTIC CURVES

ASRA ALI AND NITYA MANI

## 1. INTRODUCTION

The Modularity Theorem [2] and Eichler-Shimura [10] [18] theory enable weight 2 newforms for  $\Gamma_0(N)$  with integral coefficients to be uniquely associated to every isogeny class of elliptic curves over  $\mathbb{Q}$ . Within these results are explicit methods for constructing the 2-dimensional complex lattice  $\Lambda_E$  associated to an elliptic curve  $E$  given a weight 2 newform.

Throughout the paper, fix the weight 2 newform associated to an elliptic curve  $E/\mathbb{Q}$  as

$$(1.1) \quad f_E(z) = \sum_{n=1}^{\infty} a_E(n)q^n; \quad q = \exp(2\pi iz); \quad z \in \mathcal{H},$$

where  $\mathcal{H}$  is the upper half-plane. Here we denote the complex analytic realization of an elliptic curve as  $\mathbb{C}/\Lambda_E$ , and its modular parametrization as  $\phi_E : X_0(N) \rightarrow \mathbb{C}/\Lambda_E$ . The modular degree is the degree of the map  $\phi_E$ . We consider in this paper only elliptic curves  $E$  of modular degree 1 with conductor  $N$  such that  $\text{genus}(X_0(N)) = 1$ , which restricts  $N$  to the finite set  $\{11, 14, 15, 17, 19, 21, 27, 32, 36, 49\}$ .

Our expressions make use of the Weierstrass mock modular form associated to  $E$  (a more detailed exposition can be found in Section 2.2). These functions were recently introduced by Alfes, Bringmann, Griffin, Guerzhoy, Ono, and Rolin in [1, 12] and arise from the Weierstrass  $\zeta$ -function. Recall that the classical description of an elliptic curve,  $E \simeq \mathbb{C}/\Lambda_E$ , for some 2-dimensional complex lattice  $\Lambda_E$ . Then the *Weierstrass  $\zeta$ -function* is defined by

$$(1.2) \quad \zeta(\Lambda_E; z) = \frac{1}{z} + \sum_{w \in \Lambda_E \setminus \{0\}} \left( \frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right).$$

It is related to the Weierstrass  $\wp$ -function by differentiation

$$\frac{d}{dz} \zeta(\Lambda_E; z) = -\wp(\Lambda_E; z).$$

---

*Date:* August 22, 2016.

The authors would like to thank the NSF (grant DMS-1250467) and the Emory REU (especially Dr. Mertens and Professor Ono) for their support of our research.

We consider  $\mathcal{E}_{f_E}(z)$ , the *Eichler integral* of  $f_E$ :

$$(1.3) \quad \mathcal{E}_{f_E}(z) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} q^n.$$

Note that this is essentially the antiderivative of the newform  $f_E$  associated to the elliptic curve  $E$ .

Although the Weierstrass  $\wp$ -function is doubly periodic with respect to  $\Lambda_E$ , the  $\zeta$ -function  $\zeta(\Lambda_E; z)$  is not. Motivated by this, Eisenstein [15] constructed a modification of  $\zeta(\Lambda_E; z)$ , a lattice invariant, non-holomorphic function  $\mathfrak{Z}_E(z)$ , defined by

$$(1.4) \quad \mathfrak{Z}_E(z) = \zeta(\Lambda_E; z) - S(\Lambda_E)z - \frac{\pi}{\text{vol}(\Lambda_E)}\bar{z},$$

where

$$(1.5) \quad S(\Lambda_E) = \lim_{s \rightarrow 0} \sum_{w \in \Lambda_E \setminus \{0\}} \frac{1}{w^2 |w|^{2s}}.$$

This value  $S(\Lambda_E)$  is essentially the weight 2 Eisenstein series for the lattice  $\Lambda_E$ . Using this, we define the function  $\widehat{\mathfrak{Z}}_E(z)$  as the evaluation of  $\mathfrak{Z}_E(z)$  at the holomorphic Eichler integral  $\mathcal{E}_{f_E}(z)$ ,

$$(1.6) \quad \widehat{\mathfrak{Z}}_E(z) = \mathfrak{Z}_E(\mathcal{E}_{f_E}(z)).$$

Then  $\widehat{\mathfrak{Z}}_E(z)$  is a Harmonic Maaß form (see Section 2.2) and can be written as a sum of a holomorphic and non-holomorphic part

$$\widehat{\mathfrak{Z}}_E(z) = \widehat{\mathfrak{Z}}_E^+(z) + \widehat{\mathfrak{Z}}_E^-(z).$$

We refer to the holomorphic function  $\widehat{\mathfrak{Z}}_E^+(z)$  as the *Weierstrass mock modular form* associated to  $E$ . This follows from the proof that  $\widehat{\mathfrak{Z}}_E(z)$  is a weight 0 harmonic Maaß form since  $\widehat{\mathfrak{Z}}_E^+(z)$  is holomorphic when the modular degree is 1 (see the work of Alfes, Griffin, Ono and Rolin in [1]).

The Hasse-Weil  $L$ -function  $L(E, s) = L(f_E, s)$  plays a central role in the arithmetic of  $E$ . Indeed, the Birch and Swinnerton-Dyer Conjecture is the assertion that the arithmetic invariants of  $E/\mathbb{Q}$  are encoded by the analytic behavior of  $L(E, s)$  at  $s = 1$ . There are further  $L$ -function associated to  $E$ . Here, we consider the *shifted convolution*  $L$ -functions evaluated at  $s = 1$  defined by

$$(1.7) \quad D_{f_E}(h; s) = \sum_{n=1}^{\infty} a_E(n+h) \overline{a_E(n)} \left( \frac{1}{(n+h)^s} - \frac{1}{n^s} \right).$$

For convenience, we denote the generating function of these values in  $h$ -aspect by

$$(1.8) \quad \mathbb{L}_{f_E}(z) = \sum_{h=1}^{\infty} D_{f_E}(h; 1) q^h.$$

Motivated by the problem of numerically computing these values and understanding the explicit construction of the Weierstrass mock modular form, we offer a closed formula for these generating functions  $\mathbb{L}_{f_E}(z)$  for newforms associated to elliptic curves  $E$  of modular degree 1 and conductor  $N$  such that  $\text{genus}(X_0(N)) = 1$ . To make this precise, let  $F_{N,2}^{\rho(i)}$  be the Eisenstein series for  $\Gamma_0(N)$  nonvanishing and normalized to be 1 only at the cusp  $\rho(i)$  and vanishing at all other cusps (as in Section 2.5).

Throughout this paper, for any function  $f$ , we let  $f[h]$  denote the coefficient of  $q^h$  in the Fourier expansion of  $f$ , and recall that we let  $f_E = \sum_{n=1}^{\infty} a_E(n)q^n$  denote the weight 2 newform associated to the elliptic curve  $E$ .

For the first result, we restrict to the case where  $N$  is squarefree, restricting to the case where  $N$  is in the set  $\{11, 14, 15, 17, 19, 21\}$ .

**Theorem 1.1.** *Assume the notation and hypotheses above. Then, we have that*

$$\mathbb{L}_{f_E}(z) = \frac{\text{vol}(\Lambda_E)}{\pi} \left( (f_E(z) \cdot \widehat{\mathfrak{Z}}_E^+(z)) - \alpha f_E(z) - F_{N,2}^{\infty}(z) \right),$$

where

$$\alpha = (f_E \cdot \widehat{\mathfrak{Z}}_E^+)[1] - \frac{\pi}{\text{vol } \Lambda_E} \mathbb{L}_{f_E}(1; 1) - F_{N,2}^{\infty}[1].$$

If we drop the condition that  $N$  is squarefree and instead suppose that  $E$  has complex multiplication (restricting  $N$  to be in the set  $\{27, 32, 36, 49\}$ ), we can prove a stronger result.

**Theorem 1.2.** *Let  $E$  be an elliptic curve as above with complex multiplication. With  $\mathbb{L}_{f_E}(z)$  defined as above, we have*

$$\mathbb{L}_{f_E}(z) = \frac{\text{vol}(\Lambda_E)}{\pi} \left( (f_E(z) \cdot \widehat{\mathfrak{Z}}_E^+)(z) - F_{N,2}^{\infty}(z) \right).$$

These identities reveal a number of relations between weight 2 newforms and shifted convolution  $L$ -values that can be interpreted in many ways. Notably, these identities can be used to compute the shifted convolution  $L$ -values to arbitrary precision.

## 2. PRELIMINARIES

**2.1. Harmonic Maaß Forms and Mock Modular Forms.** We begin with a review of harmonic Maaß forms. These real-analytic modular forms were first introduced by Bruinier and Funke in [5]. Among many other important roles these forms play in number theory, work by Zagier (see [21]) shows that Ramanujan's examples of mock  $\theta$ -functions arise from holomorphic parts of harmonic Maaß forms. These forms are also intimately connected to our study of the holomorphic projections associated to certain elliptic curves.

Let  $\mathcal{H} = \{x + iy : x, y \in \mathbb{R}, y > 0\}$  denote the upper half-plane. Consider  $z = x + iy \in \mathcal{H}$  and as noted in the introduction, let  $q = e^{2\pi iz}$ . For any  $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in$

$\mathrm{SL}_2(\mathbb{Z})$  we abbreviate the action of  $\gamma$  on a function  $f$  by the Petersson slash operator of weight  $k$ ,

$$(f|_k\gamma)(z) = (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

Consider the congruence subgroup of level  $N$ , defined as follows:

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

**Definition 2.1.** ([5]) A *weak harmonic Maaß form* of weight  $k \in \mathbb{Z}$  on  $\Gamma_0(N)$  is a smooth function  $f$  on  $\mathcal{H}$  that satisfies the following three conditions:

- (1)  $(f|_k\gamma)(z) = f(z)$  for all  $\gamma \in \Gamma_0(N)$  (i.e.  $f$  transforms like a modular form on  $\Gamma_0(N)$ ),
- (2)  $\Delta_k f(z) \equiv 0$ , where if  $z = x + iy$ ,  $\Delta_k = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 2iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$
- (3)  $f(z)$  has poles at most at the cusps of  $\Gamma_0(N)$  (i.e. there exists some polynomial  $P(z)$ , such that  $f(z) - P(q^{-1}) = O(e^{-\alpha y})$  as  $y \rightarrow \infty$  and similar is true for the other cusps of  $\Gamma_0(N)$ ).

Note that hereafter, we will simply term these weight  $k$  harmonic Maaß forms for  $\Gamma_0(N)$ . Following immediately from the definition above, we see that a harmonic Maaß form admits a Fourier expansion.

**Proposition 2.2.** A weight  $k$  harmonic Maaß form  $f(z)$  admits a Fourier expansion of the form,

$$f(z) = f^+(z) + f^-(z);$$

$$f^+(z) = \sum_{n \gg -\infty} c_f^+(n) q^n; \quad f^-(z) = \sum_{n=1}^{\infty} c_f^-(n) q^n \Gamma(k-1, 4\pi n y).$$

Here,  $\Gamma(\alpha, \beta) = \int_{\beta}^{\infty} e^{-t} t^{\alpha-1} dt$  denotes the incomplete Gamma function, and  $f^+(z)$  and  $f^-(z)$  are the holomorphic and non-holomorphic parts of  $f(z)$  respectively.

This construction also gives rise to a characterization of mock modular forms in terms of harmonic Maaß forms:

**Definition 2.3.** Consider  $f(z) = f^+(z) + f^-(z)$  as defined above. If  $f^-(z) \neq 0$ , then  $f^+(z)$  is called a *mock modular form*.

We define a differential operator (as in [15]) to help characterize  $f^-(z)$  when it is nontrivial. Denote by  $H_k(\Gamma)$  the space of weight  $k$  harmonic Maaß forms for some congruence subgroup  $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ .

**Proposition 2.4.** Define a differential operator  $\xi_{2-k} : H_{2-k}(\Gamma) \rightarrow S_k(\Gamma)$  where  $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$  as

$$\xi_{2-k}(f(z)) = 2iy^k \frac{\partial f}{\partial \bar{z}}.$$

Then  $\xi_{2-k}$  is a well-defined, surjective, antilinear map with kernel  $M_{2-k}^!(\Gamma)$ , the space of weakly holomorphic weight  $2-k$  modular forms for  $\Gamma$ . Further,

$$\xi_{2-k}f(z) = -(4\pi)^{k-1} \sum_{n=1}^{\infty} c_f^-(n)q^n.$$

We term  $\sum_{n=1}^{\infty} c_f^-(n)q^n$  the *shadow* of  $f(z)$  or  $f^+(z)$ . When the shadow is non-trivial, we retrieve some conditions on the cusp behavior of  $f(z)$ .

**Proposition 2.5.** (*Lemma 2.3 of [4]*) *If  $f(z) \in H_{2-k}(\Gamma_0(N))$  has the property that  $\xi_{2-k}(f) \neq 0$ , then the principal part of  $f(z)$  is nonconstant for at least one cusp.*

**2.2. Weierstrass Mock Modular Forms.** We begin the construction of the Weierstrass mock modular form associated with an elliptic curve  $E$  of conductor  $N$ , which will prove very useful to our analysis. This Weierstrass mock modular form was introduced in [12] as a mechanism by which to understand properties of elliptic curves and their associated newforms through the language of harmonic Maaß forms. The Weierstrass mock modular form has been one of the primary objects of study by many recently, as in [1, 8].

The theory of elliptic curves gives rise to a notable example of a weight 0 harmonic Maaß form. Recall that  $E \simeq \mathbb{C}/\Lambda_E$  where  $\Lambda_E$  is a 2 dimensional complex lattice. Recall the Weierstrass  $\zeta$ -function defined in (1.2). Although it is not elliptic, its derivative is negative the Weierstrass  $\wp$ -function. This relation gives the Laurent expansion of  $\zeta$ :

**Proposition 2.6.** [9] *The Laurent expansion of  $\zeta$  is*

$$\zeta(\Lambda_E; z) = \frac{1}{z} - \sum_{k=1}^{\infty} G_{k+2}(\Lambda_E) z^{2k+1},$$

where  $G_k(\Lambda_E)$  is the Eisenstein series of weight  $k$  associated to a lattice  $\Lambda_E$ .

Again, recall the construction of the Weierstrass mock modular form given in (1.6):

$$\mathfrak{Z}_E^+(\mathcal{E}_{f_E}(z)) = \zeta(\Lambda_E; \mathcal{E}_{f_E}(z)) - S(\Lambda_E)\mathcal{E}_{f_E}(z).$$

The following theorem outlines some important properties about the Weierstrass mock modular form.

**Theorem 2.7.** ([12]) *Assume the notation and hypotheses above. Then*

- (1) *The holomorphic part  $\widehat{\mathfrak{Z}}_E^+(z) = \zeta(\Lambda_E; \mathcal{E}_f(z)) - S(\Lambda_E)\mathcal{E}_f(z)$  has poles exactly when  $\mathcal{E}_{f_E}(z)$  is a lattice point.*
- (2) *If  $\widehat{\mathfrak{Z}}_E^+(z)$  has poles in the upper half plane, there is a canonical meromorphic modular function  $M_E(z)$  such that  $\widehat{\mathfrak{Z}}_E^+(z) - M_E(z)$  is holomorphic on  $\mathcal{H}$ .*

- (3)  $\widehat{\mathfrak{Z}}_E(z) - M_E(z)$  is a harmonic Maaß form of weight 0 on  $\Gamma_0(N)$  and  $\xi_0(\widehat{\mathfrak{Z}}_E^+(z) - M_E(z)) = -(4\pi) \cdot f_E(z)$ . In particular,  $\widehat{\mathfrak{Z}}_E^+(z) - M_E(z)$  is a weight 0 mock modular form.

In particular,  $\widehat{\mathfrak{Z}}_E^+(z) - M_E(z)$  is called the Weierstrass mock modular form of  $E$ .

The kernel of the surjective  $\xi_{2-k}$  operator defined above is infinite dimensional. Selecting a suitable class of harmonic Maaß forms to serve as preimages under this  $\xi$  operator depends intimately on the following notion:

**Definition 2.8.** A harmonic weak Maaß form  $F(z) \in H_{2-k}(\Gamma_0(N))$  is *good* for  $f(z) \in S_k(\Gamma_0(N))$  if it satisfies the following 3 conditions:

- (1) The principal part of  $F(z)$  at the cusp  $\infty$  is in  $\mathbb{F}_f[q^{-1}]$ , where  $\mathbb{F}_f$  is the field obtained by adjoining the Fourier coefficients of  $f$  to  $\mathbb{Q}$ .
- (2) The principal part of  $F(z)$  at all other inequivalent cusps of  $\Gamma_0(N)$  is constant
- (3)  $\xi_{2-k}(F(z)) = \|f(z)\|^{-2}f(z)$  where  $\xi_{2-k}$  is the differential operator defined in Proposition 2.4.

This definition enables us to state the following result:

**Proposition 2.9** ([14]). *The weak harmonic Maaß form  $\widehat{\mathfrak{Z}}_E(z) - M_E(z)$ , as defined above, is good for  $f_E(z)$ , where  $f_E(z)$  is the weight 2 newform associated to  $E$ .*

In particular, we will be interested in the case where  $\widehat{\mathfrak{Z}}_E^+(z)$  itself is a mock modular form of weight 0. This arises when one can choose  $M_E(z)$  to be identically zero. The following proposition gives sufficient conditions for this to occur.

**Lemma 2.10.** *Let  $E$  be an elliptic curve of conductor  $N$  with modular parametrization  $\phi_E : X_0(N) \rightarrow E$ . If the modular degree  $\deg(\phi_E)$  is 1, then  $\widehat{\mathfrak{Z}}_E^+(z)$  does not have poles in the upper half-plane.*

*Proof.* Recall the mock modular form  $\widehat{\mathfrak{Z}}_E^+(z)$  is given by the Fourier expansion in 1.6:

$$\begin{aligned} \widehat{\mathfrak{Z}}_E^+(z) &= \widehat{\mathfrak{Z}}_E(\mathcal{E}_f(z)) \\ &= \frac{1}{\mathcal{E}_f(z)} - \sum_{k=1}^{\infty} G_{2k+2}(\Lambda_E) \mathcal{E}_f(z)^{2k+1} - S(\Lambda_E) \mathcal{E}_f(z). \end{aligned}$$

The Eichler integral  $\sum_{n=1}^{\infty} \frac{a_E(n)}{n} q^n$  is holomorphic on the upper half-plane, so it suffices to show that  $\mathcal{E}_f(z)$  does not vanish for any  $z \in \mathcal{H}$ . The modular parametrization  $\phi : X_0(N) \rightarrow \mathbb{C}/\Lambda_E$  is induced from the map  $\phi_1 : \mathcal{H} \rightarrow \mathbb{C}$  that can be given by

$$\begin{aligned} \phi_1(z) &= -2\pi i \int_z^{i\infty} f(\tau) d\tau \\ &= \sum_{n=1}^{\infty} \frac{a_E(n)}{n} q^n = \mathcal{E}_f(z). \end{aligned}$$

If the modular degree  $\deg(\phi)$  is 1 (requiring that the genus of  $X_0(N)$  is 1, since  $\deg(\phi_E) \geq \text{genus}(X_0(N))$ ), then the map  $\phi$  is an isomorphism. Thus,  $\mathcal{E}_f(z)$  does not vanish for any  $z \in \mathcal{H}$ .  $\blacksquare$

Moreover,  $\widehat{\mathfrak{Z}}_E^+(z)$  has rational Fourier coefficients if  $E$  has complex multiplication (see Theorem 1.3 in [6]).

**2.3. Maaß-Poincaré Series.** The modular parametrization of an elliptic curve  $E$  is given by a map  $\phi_E : X_0(N) \rightarrow E$  where  $X_0(N)$  is the compactification of the curve  $\Gamma_0(N) \setminus \mathcal{H}$ . Let  $f_E$  be the weight 2 newform associated to this parametrization. The Petersson norm of  $f_E$  is then

$$\|f_E\|^2 = \langle f_E, f_E \rangle = \int_{z \in \Gamma_0(N) \setminus \mathcal{H}} |f(z)|^2 dx \wedge dy.$$

Using Petersson norms, we can relate the degree of  $\phi_E$  with the area of the fundamental parallelogram of the period lattice  $\Lambda_E$ , which is the volume of the elliptic curve,  $\text{vol}(\Lambda_E)$ .

**Proposition 2.11** ([20]). *The volume  $\text{vol}(\Lambda_E)$  of an elliptic curve  $E$  is*

$$\text{vol}(\Lambda_E) = \frac{4\pi^2 \|f_E\|^2}{\deg(\phi_E)}.$$

The Petersson inner product can also be used to extract Fourier coefficients of cusp forms through Poincaré series.

A generic index  $m$  Poincaré series is given by

$$(2.1) \quad \mathbb{P}(m, k, \phi_m, N; z) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} (\phi_m^*|_k \gamma)(z),$$

where  $\Gamma_\infty = \{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z}\}$  and  $\phi_m^*(z) = \phi_m(y) \exp(2\pi i m x)$  for a function  $\phi_m : \mathbb{R}_{>0} \rightarrow \mathbb{C}$  which satisfies  $\phi_m(y) = O(y^\alpha)$  as  $y \rightarrow 0$  for some  $\alpha \in \mathbb{R}$ .

Using this, the classical index  $m$  Poincaré series  $P(m, k, N; z)$  and the Maaß-Poincaré series  $Q(-m, k, N; z)$  are defined as

$$(2.2) \quad P(m, k, N; z) = \mathbb{P}(m, k, \exp(-my), N; z)$$

$$(2.3) \quad Q(-m, k, N; z) = \mathbb{P}(-m, 2 - k, N, \mathcal{M}_{1-\frac{k}{2}}(-4\pi my); z)$$

where  $\mathcal{M}_s(y)$  is defined in terms of the  $M$ -Whittaker function

$$\mathcal{M}_s(y) = |y|^{-\frac{k}{2}} M_{\frac{k}{2} \text{sgn}(y), s-\frac{1}{2}}(|y|).$$

We can characterize a set of Maaß-Poincaré series in terms of the above functions, with a Fourier expansion given using Bessel functions and the Kloosterman sum  $K(m, n; c)$  we recall below:

$$K(m, n; c) = \sum_{d \bmod c; (c, d)=1} \exp\left(2\pi i \frac{m\bar{d} + nd}{c}\right),$$

where  $\bar{d}$  is the multiplicative inverse of  $d$  modulo  $c$ . The Fourier expansion and behavior of the Poincaré series at the cusps of  $\Gamma_0(N)$  is explained by the following proposition.

**Proposition 2.12.** (6.2 in [6], 3.3 in [16]) *If  $k \in 2\mathbb{N}$ , and  $m, N \geq 1$ , then  $Q(-m, k, N; z) \in H_{2-k}(\Gamma_0(N))$ , and has Fourier expansion of the form*

$$Q(-m, k, N; z) = Q^+(-m, k, N; z) + Q^-(-m, k, N; z),$$

where

$$Q^+(-m, k, N; z) = q^{-m} + \sum_{n=0}^{\infty} b_Q(-m, k, N; n) q^n$$

and for integers  $n \geq 0$  we have

$$b_Q(-m, k, N; n) = -2\pi(-1)^{k/2} \cdot \sum_{\substack{c>0 \\ c \equiv 0 \pmod{N}}} \left(\frac{m}{n}\right)^{\frac{k-1}{2}} \frac{K(-m, n, c)}{c} \cdot I_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right),$$

$$b_Q(-m, k, N; 0) = -\frac{2^k \pi^k (-1)^{\frac{k}{2}} m^{k-1}}{(k-1)!} \cdot \sum_{\substack{c>0 \\ c \equiv 0 \pmod{N}}} \frac{K(-m, 0, c)}{c^k}.$$

Additionally, the principal part at all other cusps is zero.

The classical Poincaré series  $P(m, k, N; z)$  and the Maaß-Poincaré series are also related by the differential operator  $\xi_{2-k}$  in the following proposition.

**Proposition 2.13.** (2.6 in [14]) *If  $k \geq 2$  is even and  $m, N \geq 1$ , then*

$$\xi_{2-k}(Q(-m, k, N; z)) = (4\pi)^{k-1} m^{k-1} (k-1) \cdot P(m, k, N; z) \in S_k(\Gamma_0(N)).$$

We can also associate a Fourier expansion to these Poincaré series that uses  $J$ -Bessel functions and Kloosterman sums.

**Proposition 2.14** (§3 [13]). *Consider the weight  $k$  Poincaré series of index  $m$  and level  $N$ ,  $P(m, k, N; z)$ . Then this Poincaré series has a Fourier expansion as  $P(m, k, N; z) = \sum_{n=0}^{\infty} b_P(m, k, N; n) q^n$  where  $b_P(m, k, N; n)$  can be defined as follows (when  $k \equiv 0 \pmod{2}$ ):*

$$b_P(m, k, N; n) = \left(\frac{n}{m}\right)^{(k-1)/2} \left( \delta_{m,n} + 2\pi i^{-k} \sum_{c>0; N|c} J_{k-1}\left(\frac{2\pi\sqrt{mn}}{c}\right) \frac{K(m, n; c)}{c} \right).$$



**2.4. Holomorphic Projection and Shifted Convolution Dirichlet Series.** The study of holomorphic projection is motivated by a desire to transform a general modular form into a holomorphic modular form such that linear functionals and the Petersson inner product are respected. The holomorphic projection was first introduced in [19] by Sturm and further developed in the work of Gross and Zagier in [11]. We will use it here to give a closed-form algebraic characterization of the shifted convolution  $L$ -series values for some modular forms associated to elliptic curves. Our approach follows previous work as in [1, 3, 12, 14].

We define the holomorphic projection for continuous functions  $f : \mathcal{H} \rightarrow \mathbb{C}$ , where  $f(z) = \sum_{n \in \mathbb{Z}} a(n, y) q^n$  that transform like a modular form of weight  $k \geq 2$  for  $\Gamma_0(N)$  and has moderate growth at the cusps. We can make this idea more precise. Suppose the cusps of  $\Gamma_0(N)$  are  $\rho(i)$  (where  $\rho(1)$  is chosen to be the infinite cusp), and consider  $\sigma_i \in \Gamma_0(N)$  so that  $\sigma_i \infty = \rho(i)$ . Then  $f$  has *moderate growth at the cusps* if for  $n > 0$

$$a(n, y) = O(y^{2-k}), \quad y \rightarrow 0,$$

and

$$f|_k \sigma_i = c_0^{(i)} + O(e^{-\alpha y}), \quad y \rightarrow \infty.$$

**Definition 2.15** ([19]). Consider a continuous function  $f(z) = \sum_{n \in \mathbb{Z}} a(n, y) q^n$  as above (where  $z = x + iy$ ) that transforms like a modular form of weight  $k \geq 2$  for  $\Gamma_0(N)$  and has moderate growth at the cusps. Then, the *holomorphic projection* of  $f(z)$ , denoted  $\pi_{\text{hol}}(f)(z)$  is constructed as follows:

$$\pi_{\text{hol}}(f)(z) = c_0^{(1)} + \sum_{n=1}^{\infty} c(n) q^n;$$

$$c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^{\infty} a(n, y) e^{-4\pi n y} y^{k-2} dy.$$

The holomorphic projection of a function satisfies several natural properties that arise from its construction:

**Proposition 2.16** ([11]). *Consider  $f$  as defined above. Then the holomorphic projection of  $f$  satisfies the following three properties.*

- (1) *If  $f$  is a holomorphic modular form,  $\pi_{\text{hol}}(f) = f$*
- (2) *If  $k > 2$ ,  $\pi_{\text{hol}}(f) \in M_k(\Gamma_0(N))$ , the space of weight  $k$  modular forms for  $\Gamma_0(N)$ . If  $k = 2$ ,  $\pi_{\text{hol}}(f) \in M_2(\Gamma_0(N)) \oplus \mathbb{C}E_2 = \tilde{M}_2(\Gamma_0(N))$ , the space of weight 2 quasi-holomorphic modular forms for  $\Gamma_0(N)$ .*
- (3)  *$\langle g, f \rangle = \langle g, \pi_{\text{hol}}(f) \rangle$  for any  $g \in S_k(\Gamma_0(N))$ .*

We can understand the holomorphic projection more explicitly in specific cases, such as the following product of a harmonic Maaß form and a cusp form.

**Proposition 2.17** ([14]). *Let  $M_{f_1}$  be the weight  $2 - k$  harmonic Maaß form whose shadow is  $f_1 \in S_k(\Gamma_0(N))$  where  $f_1(z) = \sum_{n=1}^{\infty} a_1(n)q^n$ , so  $\xi_{2-k}M_{f_1} = -(4\pi)^{k-1}f_1$ . Consider also the weight 2 cusp form for  $\Gamma_0(N)$   $f_2(z) = \sum_{n=1}^{\infty} a_2(n)q^n$ . Then,*

(2.4)

$$\pi_{hol}(M_{f_1} \cdot f_2)(z) = M_{f_1}^+(z) \cdot f_2(z) - (k-2)! \sum_{h=1}^{\infty} \left[ \sum_{n=1}^{\infty} a_2(n+h) \overline{a_1(n)} \left( \frac{1}{(n+h)^{k-1}} - \frac{1}{n^{k-1}} \right) \right] q^h.$$

Now consider a strong Weil curve  $E$  with associated weight 2 newform  $f_E$  for  $\Gamma_0(N)$ , where the genus of the modular curve  $X_0(N)$  is 1, and let  $\widehat{\mathfrak{Z}}_E$  be defined as in (1.6). Then, we can compute the holomorphic projection of  $f_E \cdot \widehat{\mathfrak{Z}}_E$  as follows:

**Lemma 2.18.** *Let  $E$  be a strong Weil curve with associated weight 2 newform  $f_E$  for  $\Gamma_0(N)$ , where the genus( $X_0(N)$ ) = 1 and  $\deg(\phi_E) = 1$ , and let  $\widehat{\mathfrak{Z}}_E$  be defined as in (1.6). Then, we have the following:*

$$\pi_{hol}(f_E \cdot \widehat{\mathfrak{Z}}_E)(z) = \frac{\text{vol}(\Lambda_E)}{\pi} f_E(z) \widehat{\mathfrak{Z}}_E^+(z) - \sum_{h=1}^{\infty} D_{f_E}(h; 1) q^h.$$

*Proof.* Since  $\dim(S_2(\Gamma_0(N))) = \text{genus}(X_0(N)) = 1$ , the modular form  $f_E$  is a scalar multiple of the Poincaré series  $f_E = \frac{1}{\beta} P(1, 2, N; z)$ , where  $P(m, k, N; z)$  is the Poincaré series described in Proposition 2.14. Then the Petersson coefficient formula (§3 [13]) and Proposition 1.5 yields  $\beta = \frac{\pi}{\text{vol}(\Lambda_E)}$ . Following the computation in Corollary 1.2 of [14], we obtain the holomorphic projection in terms of the Poincaré series

$$\pi_{hol}(f_E \cdot \widehat{\mathfrak{Z}}_E)(z) = P(1, 2, N; z) \widehat{\mathfrak{Z}}_E^+(z) - \sum_{h=1}^{\infty} \mathbb{L}_{f_E}[h] q^h.$$

We can apply Proposition 2.17 with  $P(1, 2, N; z) = \frac{1}{\beta} f_E$ , which yields the Lemma.  $\blacksquare$

*Remark.* For the remainder of the paper, we will define a new function  $\widehat{\pi}_{hol}$ , a scalar multiple of the holomorphic projection by the constant  $\frac{\pi}{\text{vol}(\Lambda_E)}$  for ease of algebraic computation and numerical characterization. Thus, we will say

$$\widehat{\pi}_{hol}(f_E \cdot \widehat{\mathfrak{Z}}_E)(z) = f_E(z) \widehat{\mathfrak{Z}}_E^+(z) - \frac{\pi}{\text{vol}(\Lambda_E)} \sum_{h=1}^{\infty} D_{f_E}(h; 1) q^h.$$

**2.5. Eisenstein Series.** In order to understand  $\widehat{\pi}_{hol}(f_E \cdot \widehat{\mathfrak{Z}}_E)$ , we define a basis for the space of *weight 2 quasimodular forms* for  $\Gamma_0(N)$ , the space  $\mathcal{E}_2(\Gamma_0(N)) \oplus \mathbb{C}E_2$ , denoted  $\widetilde{\mathcal{E}}_2(\Gamma_0(N))$ . To do this, we follow the construction given in §2 of Chapter VII in [17] and arrive at a set of forms  $F_{N,2}^{-a_2/a_1}$  described below:

**Proposition 2.19.** *There is a set of weight 2 quasimodular Eisenstein forms for  $\Gamma_0(N)$ , say  $F_{N,2}^{-a_2/a_1}$ , such that if  $(a_1, a_2) = 1$ , then  $F_{N,2}^{-a_2/a_1}$  is 1 at the cusps  $\Gamma_0(N)$ -equivalent to  $-a_2/a_1$  and 0 at the other cusps. The set of linearly independent  $F_{N,2}^{-a_2/a_1}$  forms a basis for  $\tilde{\mathcal{E}}_2(\Gamma_0(N))$ .*

Using these Eisenstein series, we can give an alternate representation of the holomorphic projection described in the previous section.

**Lemma 2.20.** *Consider a strong Weil curve  $E$  with associated modular form  $f_E$  and  $\hat{\mathfrak{Z}}_E$  as defined in 1.6. Then,*

$$(2.5) \quad \hat{\pi}_{hol}(f_E \cdot \hat{\mathfrak{Z}}_E) = \alpha f_E + \sum_i \beta_i F_{N,2}^{\rho(i)}, \quad \beta_i \in \mathbb{C}$$

where  $F_{N,2}^{\rho(i)}$  is the weight 2 quasimodular form for  $\Gamma_0(N)$  that takes the value 1 at the cusp  $\rho(i)$  and vanishes at all other inequivalent cusps.

### 3. PROOFS OF THEOREMS

Throughout this sections, let  $E$  be a strong Weil elliptic curve with conductor  $N_E$  and associated weight 2 newform  $f_E = \sum_{n=1}^{\infty} a_E(n)q^n$  of level  $N = N_E$ , where  $\text{genus}(X_0(N)) = 1$ .

**Lemma 3.1.** *Consider  $E$  as defined above. Then  $\hat{\mathfrak{Z}}_E^+$  vanishes at all cusps not equivalent to the  $\infty$  cusp of  $\Gamma_0(N)$ .*

*Proof.* If  $E$  satisfies the above conditions, then  $\hat{\mathfrak{Z}}_E$  is good for  $f_E$ . In particular,  $\hat{\mathfrak{Z}}_E$  has a pole at  $\infty$  and constant principal part at the cusps of  $\Gamma_0(N)$  by Proposition 2.9. On the other hand, the index  $-1$  Maaß-Poincaré series  $Q(-1, 2, N; z)$  has a pole at  $\infty$  and zero principal part at the cusps (2.12). The difference  $\hat{\mathfrak{Z}}_E(z) - Q(-1, 2, N; z)$  is a weight 0 harmonic Maass form with no poles and constant value at each of the cusps. The differential operator  $\xi_0$  maps harmonic Maass forms of weight  $2 - k$  to cusp forms of weight  $k$ . Then since the dimension of  $S_2(\Gamma_0(N))$  is 1, we have

$$\xi_0(\hat{\mathfrak{Z}}_E(z) - Q(-1, 2, N; z)) = c \cdot f_E(z)$$

for some constant  $c$ . However, by Lemma 2.5, since  $\hat{\mathfrak{Z}}_E(z) - Q(-1, 2, N; z)$  has constant principal part at all cusps,  $c = 0$ . This implies that the difference is holomorphic, and so is constant. Then  $\hat{\mathfrak{Z}}_E(z)$  and  $Q(-1, 2, N; z)$  are equal up to an additive constant since both have leading term  $q^{-1}$ , as are their holomorphic parts  $\hat{\mathfrak{Z}}_E^+(z)$  and  $Q^+(-1, 2, N; z)$ . Thus, as in [16], since  $Q^+(-1, 2, N; z)$  vanishes at all non-infinity cusps by Lemma 2.12,  $\hat{\mathfrak{Z}}_E^+(z)$  vanishes at all non-infinite cusps.  $\blacksquare$

Using this Lemma, we are able to prove the first theorem.

*Proof of Theorem 1.1.* From Lemma 2.18 and Lemma 2.20, we obtain that

$$\widehat{\pi}_{\text{hol}}(\widehat{\mathfrak{Z}}_E \cdot f_E)(z) = f_E(z) \cdot \widehat{\mathfrak{Z}}_E^+(z) - \frac{\pi}{\text{vol}(\Lambda_E)} \sum_{h=1}^{\infty} D_{f_E}(h; 1) q^h = \alpha f_E + \sum_i \beta_i F_{N,2}^{\rho(i)}(z).$$

We can compute the holomorphic projection at each cusp  $\rho(i)$  gives the value of  $\beta_i$ . Note that the  $L$ -series generating function  $\mathbb{L}_{f_E}(z)$  and  $f_E$  both vanish at all cusps  $\rho(i)$ . Applying Lemma 3.1,  $\widehat{\mathfrak{Z}}_E^+$  vanishes at all cusps inequivalent to the  $\infty$  cusp of  $\Gamma_0(N)$ . Consequently the holomorphic projection vanishes at all cusps not  $\Gamma_0(N)$ -equivalent to  $\infty$  and is 1 at the cusp  $\infty$ . If we let  $\rho(1)$  denote the  $\infty$  cusp,  $\beta_i = 0$  for  $i \neq 1$  and  $\beta_1 = 1$ . Using this, we can compute  $\alpha$  by equating the first Fourier coefficient of both expressions for  $\widehat{\pi}_{\text{hol}}$ . Rearranging gives the desired expression for  $\mathbb{L}_{f_E}(z)$ . ■

We also consider the case where  $E$  as defined in the beginning has complex multiplication, which will not only give an analogous result for the shifted convolution  $L$ -series values, but it will also give a recursion on the coefficients of the modular form  $f_E(z)$  that can be written in terms of  $G_4$  and  $G_6$ , an explicit example of the Modularity Theorem.

Now, we can prove Theorem 1.2.

*Proof of Theorem 1.2.* Following the proof for Theorem 1.1, we obtain that  $\beta_1 = 1$ , and  $\beta_i = 0$  for  $i \neq 1$  in Equation (2.5). Since  $f_E$  is only supported at coefficients 1 mod  $p$ , where the left hand side is not supported (see Lemma 3.3),  $\alpha = 0$ . Thus

$$\widehat{\pi}_{\text{hol}}(\widehat{\mathfrak{Z}}_E \cdot f_E)(z) = F_{N,2}^{\infty}(z).$$

Then, we can compute the desired closed form expression for the shifted-convolution  $L$ -series values:

$$\mathbb{L}_{f_E}(z) = \frac{\text{vol}(\Lambda_E)}{\pi} ((f_E(z) \cdot \widehat{\mathfrak{Z}}_E^+(z)) - F_{N,2}^{\infty}(z)).$$
■

**Lemma 3.2.** *Consider  $E$  as defined above which also has complex multiplication and thus conductor  $N \neq 49$  with associated modular form  $f_E(z)$ . Then  $D_{f_E}(h; 1) = 0$  if  $h \not\equiv 0 \pmod{n_0}$ .*

*Proof.* Suppose that  $E$  has complex multiplication. Then if  $p$  is a prime inert in the CM field,  $a_E(p) = 0$ . Since  $f_E$  is a weight 2 newform, its coefficients are multiplicative. Thus, as in §5 of [6],  $a_E(n) = 0$  for all  $n \not\equiv 1 \pmod{n_0}$  where  $n_0|N$  is a curve-dependent value always at least 3. For example, in the case of the  $\Gamma_0(27)$ -optimal elliptic curve with complex multiplication,  $n_0 = 3$ . Now recall that  $D_{f_E}(h; 1)$  is defined by

$$D_{f_E}(h; 1) = \sum_{n=1}^{\infty} a_E(n+h) a_E(n) \left( \frac{1}{n+h} - \frac{1}{n} \right).$$

Suppose that  $D_{f_E}(h; 1)$  is nonvanishing. Then both  $a_E(n + h)$  and  $a_E(n)$  must be nonvanishing and thus  $n + h, n \equiv 1 \pmod{n_0}$ . This yields  $h \equiv 0 \pmod{n_0}$ . ■

*Remark.* For  $N = 49$ , the support of the Hecke eigenvalues  $a_E(n)$  are at  $n \equiv 1, 2, 4 \pmod{7}$ . In this case, the proof does not hold, and the  $L$ -series has support everywhere.

**Lemma 3.3.** *Consider  $E$  with complex multiplication as defined at the beginning of the section with conductor  $N \neq 49$  and associated modular form  $f_E$ . Then there exists some  $n_0 \geq 3$  with  $n_0 | N$  such that  $\widehat{\pi}_{\text{hol}}(f_E \cdot \widehat{\mathfrak{Z}}_E)[h] = 0$  if  $h \not\equiv 0 \pmod{n_0}$ .*

*Proof.* The derivative of the Weierstrass mock modular form  $\widehat{\mathfrak{Z}}_E^+$  for the three strong Weil curves of conductor  $N = 27, 32$ , and  $36$  are given as eta-quotients in the following table (see [8]).

$N$	$\widehat{\mathfrak{Z}}_E^+(\cdot)$	$q \frac{d}{dq}(\widehat{\mathfrak{Z}}_E^+)$
27	$q^{-1} + \frac{1}{2}q^2 + \frac{1}{5}q^5 + \frac{3}{4}q^8 + \dots$	$-\frac{\eta(3\tau)\eta(9\tau)^6}{\eta(27\tau)^3}$
32	$q^{-1} + \frac{2}{3}q^3 + \frac{1}{7}q^7 - \frac{2}{11}q^{11} + \dots$	$-\frac{\eta^2(4\tau)\eta^6(16\tau)}{\eta^4(32\tau)}$
36	$q^{-1} + \frac{3}{5}q^5 + \frac{1}{11}q^{11} + \dots$	$-\frac{\eta^3(6\tau)\eta(12\tau)\eta(18\tau)}{\eta^3(36\tau)}$

The support of the derivative  $q \frac{d}{dq}(\widehat{\mathfrak{Z}}_E^+)$  is the same as the support of the Weierstrass mock modular form. The form of the eta-quotients indicates that the support for  $\widehat{\mathfrak{Z}}_E^+$  for  $N = 27$  is  $-1 \pmod{3}$ , for  $N = 32$  is  $-1 \pmod{4}$ , and for  $N = 36$  is  $-1 \pmod{6}$ . Let  $n_0 = 3, 4, 6$  for  $N = 27, 32, 36$  respectively.

Each of these curves has complex multiplication, and their associated modular forms  $f_E$  have support at  $1 \pmod{n_0}$ . Thus the product of the Weierstrass mock modular form  $\widehat{\mathfrak{Z}}_E^+$  and  $f_E$  has support at  $0 \pmod{n_0}$ .

By Lemma 3.2, the  $L$ -series is also only supported at  $0 \pmod{n_0}$ , which yields that the holomorphic projection

$$\pi_{\text{hol}}(f_E \cdot \widehat{\mathfrak{Z}}_E^+) = f_E \cdot \widehat{\mathfrak{Z}}_E^+ - \frac{\pi}{\text{vol}(\Lambda_E)} \mathbb{L}_{f_E}(z)$$

has support only at  $0 \pmod{n_0}$ . ■

*Remark.* Note that results such as the above theorems may be able to be transformed to give recurrence relations on the modular form coefficients, similar to those obtained in [7, 15].

## 4. EXAMPLES

4.1. **Conductor**  $N = 11$ . Consider the modular curve  $X_0(11)$  of dimension 1. There is a single isogeny class of elliptic curves, and the strong Weil curve is given by the Weierstrass equation

$$E : y^2 + y = x^3 - x^2 - 10x - 20.$$

Numerically, we find that  $S(\Lambda_E) = 0.38124 \dots$ . Using this and the Fourier expansion of the Weierstrass  $\zeta$ -function, the corresponding weight 0 mock modular form  $\widehat{\mathfrak{Z}}_E^+(z)$  is given by

$$q^{-1} + 1 + 0.9520 \dots q + 1.547 \dots q^2 + 0.3493 \dots q^3 + 1.976 \dots q^4 - 2.609 \dots q^5 + O(q^6).$$

Using the formula given in Proposition 2.18, one can compute the  $L$ -series numerically. Using 100000 coefficients of  $f_E$ , we have the Fourier expansion  $\mathbb{L}_{f_E}(z) = \sum_{n=1}^{\infty} D_{f_E}(h; 1) q^h$ :

$$(4.1) \quad \mathbb{L}_{f_E}(z) = -0.7063 \dots q - 1.562 \dots q^2 - 0.0944 \dots q^3 - 1.237 \dots q^4 + 2.026 \dots q^5 + O(q^6)$$

The space of weight 2 quasimodular Eisenstein series for  $\Gamma_0(11)$  is 2 dimensional, and a basis is given by  $F_{11,2}^0$  and  $F_{11,2}^\infty$ , where  $F_{11,2}^\infty$  is

$$F_{11,2}^\infty = 1 + \frac{1}{5}q + \frac{3}{5}q^2 + \frac{4}{5}q^3 + \frac{7}{5}q^4 + \frac{6}{5}q^5 + \frac{12}{5}q^6 + O(q^7)$$

Then, if we take  $\alpha = .00159$ , we obtain another way to retrieve the  $L$ -series values in Equation (4.1):

$$\begin{aligned} & \frac{\text{vol}(\Lambda_E)}{\pi} \left( (f_E \cdot \widehat{\mathfrak{Z}}_E^+) - \alpha f_E - \sum \beta_{\rho(i)} F_{11,2}^{\rho(i)} \right) = \\ & -0.706 \dots q - 1.562 \dots q^2 - 0.0930 \dots q^3 - 1.234 \dots q^4 + 2.024 \dots q^5 + O(q^6) \end{aligned}$$

4.2. **Conductor**  $N = 27$ . Recall the strong Weil curve of conductor 27 given by the Weierstrass equation  $E_{27} : y^2 + y = x^3 - 7$  (Cremona label 27a1). The weight 2 modular form associated with  $E_{27}$  is given by

$$f_{E_{27}} = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + O(q^{20}).$$

Using the Fourier expansion of the Weierstrass  $\zeta$ -function, the weight 0 mock modular form associated to  $\widehat{\mathfrak{Z}}_{E_{27}}^+(z)$  is given by

$$(4.2) \quad \widehat{\mathfrak{Z}}_{E_{27}}^+(z) = q^{-1} + \frac{1}{2}q^2 + \frac{1}{5}q^5 + \frac{3}{4}q^8 - \frac{6}{11}q^{11} - \frac{1}{2}q^{14} + O(q^{17}).$$

The holomorphic projection is given by

$$\widehat{\pi}_{\text{hol}}(f_E \cdot \widehat{\mathfrak{Z}}_{E_{27}}^+)(z) = 1 + 3q^9 + 9q^{18} - 12q^{27} \dots$$

On the other hand, we find that the normalized element of the weight 2 quasimodular forms for  $\Gamma_0(27)$  that vanishes at all cusps but  $\infty$  is given by

$$F_{27,2}^\infty = 1 + 3q^9 + 9q^{18} - 12q^{27} \dots$$

This agrees with the computation for the holomorphic projection. Then the generating function  $\mathbb{L}_f(z)$  can also be computed by Theorem 1.2, giving arbitrary precision computations for these slowly convergent shifted convolution  $L$ -series values that could only previously be computed term-by-term.

## REFERENCES

- [1] C. Alfes, M. Griffin, K. Ono, and L. Rolén. Weierstrass mock modular forms and elliptic curves. *Research in Number Theory*, 1(1):1–31, 2015.
- [2] C. Breuil, B. Conrad, F. Diamond, and R. Taylor. On the modularity of elliptic curves over  $\mathbb{Q}$ : wild 3-adic exercises. *Journal of the American Mathematical Society*, 14(4):843–939, 2001.
- [3] K. Bringmann, M. H. Mertens, and K. Ono.  $p$ -adic properties of modular shifted convolution Dirichlet series. *Proceedings of the American Mathematical Society*, 144(4):1439–1451, 2016.
- [4] K. Bringmann and K. Ono. Coefficients of harmonic Maass forms. In *Partitions,  $q$ -Series, and Modular Forms*, pages 23–38. Springer, 2012.
- [5] J. H. Bruinier, J. Funke, et al. On two geometric theta lifts. *Duke Mathematical Journal*, 125(1):45–90, 2004.
- [6] J. H. Bruinier, K. Ono, and R. C. Rhoades. Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues. *Mathematische Annalen*, 342(3):673–693, 2008.
- [7] D. Choi. Poincaré series and the divisors of modular forms. *Proceedings of the American Mathematical Society*, 138(10):3393–3403, 2010.
- [8] A. Clemm. Modular forms and weierstrass mock modular forms. *Mathematics*, 4(1):5, 2016.
- [9] F. Diamond and J. Shurman. *A first course in modular forms*, volume 228. Springer Science & Business Media, 2006.
- [10] M. Eichler. Eine verallgemeinerung der abelschen integrale. *Mathematische Zeitschrift*, 67(1):267–298, 1957.
- [11] B. H. Gross and D. B. Zagier. Heegner points and derivatives of  $L$ -series. *Inventiones mathematicae*, 84(2):225–320, 1986.
- [12] P. Guerzhoy. A mixed mock modular solution of the kaneko–zagier equation. *The Ramanujan Journal*, 36(1-2):149–164, 2015.
- [13] J. Liu and Y. Ye. Petersson and Kuznetsov trace formulas. *AMS IP Studies in Advanced Mathematics*, 37:147, 2006.
- [14] M. H. Mertens and K. Ono. Special values of shifted convolution Dirichlet series. *Mathematika*, 62(01):47–66, 2016.
- [15] K. Ono. Unearthing the visions of a master: harmonic Maass forms and number theory. *Curr. Dev. Math.*, 2008:347–454, 209.
- [16] R. C. Rhoades. Linear relations among Poincaré series via harmonic weak maass forms. *The Ramanujan Journal*, 29(1-3):311–320, 2012.
- [17] B. Schoeneberg. *Elliptic modular functions: an introduction*, volume 203. Springer Science & Business Media, 2012.
- [18] P. G. Shimura. Sur les intégrales attachées aux formes automorphes. *Journal of the Mathematical Society of Japan*, 11(4):291–311, 1959.

- [19] J. Sturm. Projections of  $\mathcal{C}^\infty$  automorphic forms. *Bulletin of the American Mathematical Society*, 2(3):435–439, 1980.
- [20] D. Zagier. Modular parametrizations of elliptic curves. *Canad. Math. Bull*, 28(3):372–384, 1985.
- [21] S. Zwegers. Mock theta functions. 2003.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, DEPARTMENT OF MATHEMATICS, CAMBRIDGE, MA

*E-mail address:* `asra@mit.edu`

STANFORD UNIVERSITY, DEPARTMENT OF MATHEMATICS, STANFORD, CA 94305

*E-mail address:* `nityam@stanford.edu`